$$= \frac{\ln a}{e}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

It is known that for real x tending to zero, we have $e^x = 1 + x + O\left(x^2\right)$.

Since $\lim\limits_{n\to\infty} E_n = e$, so $\sqrt[n]{E_n} - 1 = e^{\frac{\ln E_n}{n}} - 1 = \dfrac{\ln E_n}{n} + O\left(\dfrac{1}{n^2}\right)$, and

$a^{\sqrt[n]{E_n}-1} - 1 = e^{\left(\sqrt[n]{E_n}-1\right)\ln a} - 1 = \dfrac{(\ln E_n)(\ln a)}{n} + O\left(\dfrac{1}{n^2}\right)$, where the last constant implied by O depends at most on a. Hence, by Stirling's formula

$n! = \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\left(1 + O\left(\dfrac{1}{n}\right)\right)$ as n tends to infinity, we obtain

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(a^{\sqrt[n]{E_n}-1} - 1\right) = \frac{\ln a}{e}.$$

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego, Viveiro, Spain; Moti Levy, Rehovot, Israel, and the proposers.

- **5311:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \sqrt{\left(\frac{x^2}{3} + 3y^2\right)\left(\frac{2}{xy} + \frac{1}{z^2}\right)} \geq 3\sqrt{10}.$$

Solution 1 by Arkady Alt, San Jose, CA

Since by AM-GM Inequality $\dfrac{x^2}{3} + 3y^2 = \dfrac{x^2 + 9y^2}{3} \geq \dfrac{1}{3}\cdot 10\sqrt[10]{x^2\cdot\left(y^2\right)^9} = \dfrac{10}{3}\sqrt[5]{xy^9}$ and

$\dfrac{2}{xy} + \dfrac{1}{z^2} \geq 3\sqrt[3]{\left(\dfrac{1}{xy}\right)^2\cdot\dfrac{1}{z^2}} = \dfrac{3}{\sqrt[3]{x^2y^2z^2}}$ then

$\sqrt{\left(\dfrac{x^2}{3} + 3y^2\right)\left(\dfrac{2}{xy} + \dfrac{1}{z^2}\right)} \geq \sqrt{\dfrac{10}{3}\sqrt[5]{xy^9}\cdot\dfrac{3}{\sqrt[3]{x^2y^2z^2}}} \iff \dfrac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot x^{\dfrac{1}{10}}y^{\frac{9}{10}}$ and, therefore,

using again AM-GM Inequality we obtain

$$\sum_{cyclic}\sqrt{\left(\tfrac{x^2}{3} + 3y^2\right)\left(\tfrac{2}{xy} + \tfrac{1}{z^2}\right)} \geq \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot\sum_{cyclic} x^{\dfrac{1}{10}}y^{\dfrac{9}{10}} \geq$$

$$\frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{x^{\frac{1}{10}}y^{\frac{9}{10}}\cdot y^{\frac{1}{10}}z^{\frac{9}{10}}\cdot z^{\frac{1}{10}}x^{\frac{9}{10}}} = \frac{\sqrt{10}}{\sqrt[3]{xyz}}\cdot 3\sqrt[3]{xyz} = 3\sqrt{10}.$$

Equality holds if $x = y = z$.